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Hankel Type Transformation and Convolution of Tempered Beurling Distributions

B.B.Waphare
MIT ACSC, Alandi, Tal: Khed, Dist: Pune, Maharashtra, India
balasahebwaphare@gmail.com

Abstract

In this paper we develop the distributional theory of Hankel type transformation. New Frechet function type spaces $\mathcal{H}_{\alpha,\beta}\,(w)$ are introduced. The functions in $\mathcal{H}_{\alpha,\beta}\,(w)$ have a growth in infinity restricted by the Beurling type function w. We study on $\mathcal{H}_{\alpha,\beta}\,(w)$ and its dual the Hankel type transformation and the Hankel type convolution.

Keywords: Beurling type distributions, Hankel type transformation, Hankel type convolution.

Mathematics subject classification: 46F12.

Introduction

The theory of Hankel transform have been studied by many researchers in past from time to time.

The Hankel type transformation is defined by

$$h_{\alpha,\beta}\,(\phi)\,(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\,\phi\,(y)\,y^{4\alpha}\,dy,\;\; x \in (0,\infty),$$

where $J_{\alpha-\beta}$ represents the Bessel type function of the first kind and order $\alpha-\beta$. Throughout this paper, we will assume that $(\alpha-\beta) > -\frac{1}{2}$. Notice that if ϕ is a Lebesgue measurable function on $(0,\infty)$ and

$$\int_0^\infty x^{4\alpha}\;\; |\phi\,(x)|\,dx < \infty,$$

then, since the function $z^{-(\alpha-\beta)} J_{\alpha-\beta}\,(z)$ is bounded on $(0,\infty)$, the Hankel type transform $h_{\alpha,\beta}\,(\phi)$ is a bounded function on $(0,\infty)$. Moreover, $h_{\alpha,\beta}\,(\phi)$ is continuous on $(0,\infty)$ and according to the Riemann-Lebesgue theorem for Hankel type transform ([15]),

$$\lim_{x\to\infty} h_{\alpha,\beta}\,(\phi)\,(x) = 0.$$

The study of Hankel transformation in distribution spaces was studied by Zemanian ([18],[19]). More recently Waphare and Gunjal [16] have investigated the $h_{\alpha,\beta} -$ transform of generalized functions with exponential growth. Our objective in this paper is to define the Hankel type transformation on new distribution spaces that are in a certain sense, between the spaces considered in [16] and [18].

Following Zemanian [18], we can introduce the space $\mathcal{H}_{\alpha,\beta}$ that consists of all those complex valued and smooth functions ϕ defined on $(0,\infty)$ such that, for every $m,n \in \mathbb{N}$,

$$\rho_{m,n}^{\alpha,\beta}(\phi) = Sup_{x\,\in\,(0,\infty)}(1+x^2)^m \left|\left(\tfrac{1}{x}D\right)^n \left(x^{2\beta-1}\,\phi(x)\right)\right| < \infty.$$

On $\mathcal{H}_{\alpha,\beta}$ we consider the topology generated by the family $\left\{\rho_{m,n}^{\alpha,\beta}\right\}_{m,n\,\in\,\mathbb{N}}$ of seminorms. Then $\mathcal{H}_{\alpha,\beta}$ is a Frechet space and the Hankel type transformation $H_{\alpha,\beta}$ defined by

$$H_{\alpha,\beta}\,(\phi)\,(x) = \int_0^\infty (xy)^{\alpha+\beta}\, J_{\alpha-\beta}(xy)\,\phi(y)\,dy\,, \qquad x \in (0,\infty)\,,$$

is an automorphism of $\mathcal{H}_{\alpha,\beta}$ (see [18, Lemma 8]). Note that the two forms $h_{\alpha,\beta}$ and $H_{\alpha,\beta}$ of Hankel type transforms are related through

$$H_{\alpha,\beta}\,(\phi)\,(x) = x^{2\alpha}\, h_{\alpha,\beta}\left(y^{2\beta-1}\,\phi\right)(x)\,, \qquad x \in (0,\infty)\,.$$

The Hankel type transformation $H_{\alpha,\beta}$ is defined on the dual $\mathcal{H}'_{\alpha,\beta}$ of $\mathcal{H}_{\alpha,\beta}$ by transposition. Altenburg [1] developed a theory similar to that of Zemanian for the $h_\mu -$ transformation. Note that the space $\mathcal{H}_{-1/2}$ coincides with the space $\mathcal{H}$ considered in [1].

In Waphare and Gunjal [16], the space $M_{\alpha,\beta}$ constituted by all the complex valued and smooth functions ϕ defined on $(0,\infty)$ satisfying

$$\eta_{m,n}^{\alpha,\beta}\,(\phi) = \sup_{x\,\in\,(0,\infty)} e^{mx} \left|\left(\frac{1}{x}\,D\right)^n \left(x^{2\beta-1}\phi\,(x)\right)\right| < \infty,$$

for each $m,n \in \mathbb{N}$ is considered.

In Waphare and Gunjal [16, Theorem 2.4] a characterization of the image by $H_{\alpha,\beta}$ of the space $\chi_{\alpha,\beta}$ as a certain space of entire functions with a restricted growth on horizontal strips is given. The Hankel type transform $H_{\alpha,\beta}$ is defined on the corresponding dual spaces by transposition. We introduce here the space $\mathcal{H}_{\alpha,\beta}\,(w)$ constituted by functions whose growth is restricted by e^{nw}, $n \in \mathbb{N}$, where w is a function that we will define later.

Hirschman [11], Haimo [10] and Cholewinski [7] investigated the Hankel convolution operation.

The convolution associated with the $h_{\alpha,\beta}$ −transformation is defined as follows. The Hankel type convolution $f\#_{\alpha,\beta}\, g$ of order $\alpha-\beta$ of the measurable functions $f \; and \; g$ is given through

$$\left(f\#_{\alpha,\beta}\, g\right)(x) = \int_0^\infty f(y)\left({}_{\alpha,\beta}\tau_x g\right)(y)\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy,$$

where the Hankel type translation operator ${}_{\alpha,\beta}\tau_x\, g$, $x \in (0,\infty)$, of g is defined by

$$\left({}_{\alpha,\beta}\tau_x\, g\right)(y) = \int_0^\infty g(z)\, D_{\alpha,\beta}\,(x,y,z)\,\frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dz,$$

provided that the above integrals exists. Here $D_{\alpha,\beta}$ is the following function

$$D_{\alpha,\beta}\,(x,y,z) = \left(2^{\alpha-\beta}\Gamma(3\alpha+\beta)\right)^2 \int_0^\infty (xt)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(xt)\,(yt)^{\alpha-\beta} J_{\alpha-\beta}\,(yt)$$

$$\times\,(zt)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(zt)\, t^{4\alpha}\,dt, \qquad x,y,z \in (0,\infty).$$

Moreover, we define ${}_{\alpha,\beta}\tau_o\, g = g$.

The study of the $\#_\mu$ − convolution on L_p − spaces was developed in [10] and [11].

If we denote by $L_{1,\alpha,\beta}$ the space of complex valued and measurable functions f on $(0,\infty)$ such that

$$\int_0^\infty |f(x)|\, x^{4\alpha}\, dx < \infty,$$

the following interchange formula

$$h_{\alpha,\beta}\left(f\#_{\alpha,\beta} g\right) = h_{\alpha,\beta}\,(f)\, h_{\alpha,\beta}\,(g),$$

holds for every $f, g \in L_{1,\alpha,\beta}$.

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The investigation of the distributional Hankel convolution was started by de Sousa-Pinto [13], who considered any $\mu = 0$. Betancor and Marrero ([3], [4] and [12]) studied the Hankel convolution on the Zemanian spaces. In [16], Waphare and Gunjal analyzed the $\#_{\alpha,\beta} -$ convolution of distributions with exponential growth.

In the sequel, since we think any confusion is possible, to simplify we will write $\#, \tau_x$, $x \in [0,\infty)$ and D instead of $\#_{\alpha,\beta}$, ${}_{\alpha,\beta}\tau_x,\ x \in [0,\infty)$ *and* $D_{\alpha,\beta}$ respectively.

As in [6], we consider continuous, increasing and non-negative functions w defined on $[0,\infty)$ such that $w(0) = 0, w(1) > 0$, and it satisfies the following three properties

(i) $w(x+y) \le w(x) + w(y),\ x, y \in [0,\infty)$,

(ii) $\int_1^\infty (w(x)/x^2)\, dx < \infty$, and

(iii) there exist $a \in \mathbb{R}$ and $b > 0$ such that $w(x) \ge a + b \log(1+x),\ x \in [0,\infty)$.

We say $w \in \mathcal{M}$ when w satisfies the above conditions. If w is extended to $\mathbb{R}$ as an even function, then w satisfies the subadditivity property (i) for every $x, y \in \mathbb{R}$..

Beurling [5] developed a general theory of distributions that extends the Schwartz theory. Some aspects of that theory were presented and completed by Bjorck [6]. Now we recall some definitions and properties from [2] which will be useful in the sequel.

Let $w \in \mathcal{M}$. For every $a > 0$ the space $B^a_{\alpha,\beta}(w)$ is constituted by all those complex-valued and smooth functions ϕ on $(0,\infty)$ such that $\phi(x) = 0,\ x \ge a,\ \phi$ and $h_{\alpha,\beta}(\phi) \in L_{1,\alpha,\beta}$ and that

$$\delta_n^{\alpha,\beta}(\phi) = \int_0^\infty |h_{\alpha,\beta}(\phi)(x)|\, e^{n\,w(x)}\, x^{4\alpha}\, dx < \infty,$$

for every $n \in \mathbb{N}$. $B^a_{\alpha,\beta}(w)$ is a Frechet space when we consider on it the topology generated by the system $\{\delta_n^{\alpha,\beta}\}_{n \in \mathbb{N}}$ of seminorms. It is clear that $B^a_{\alpha,\beta}(w)$ is continuously contained in $B^b_{\alpha,\beta}(w)$ when $0 < a < b$. The union space

$$B_{\alpha,\beta}(w) = \bigcup_{a>0} B^b_{\alpha,\beta}(w)$$

is endowed with the inductive topology.

For every $x \in (0,\infty)$, the Hankel type translation τ_x defines a continuous linear mapping from $B_{\alpha,\beta}(w)$ into itself. Then we can define the Hankel type convolution $T \# \phi$ of $T \in B_{\alpha,\beta}(w)'$, the dual space of $B_{\alpha,\beta}(w)$ and $\phi \in B_{\alpha,\beta}(w)$ by

$$(T\#\phi)(x) = \langle T, \tau_x \phi \rangle, \qquad x \in [0,\infty).$$

By $\mathcal{E}_{\alpha,\beta}(w)$ we denote the space of pointwise multipliers of $B_{\alpha,\beta}(w)$. $\mathcal{E}_{\alpha,\beta}(w)$ is endowed with the topology induced by the topology of pointwise convergence of the space $\mathfrak{L}\left(B_{\alpha,\beta}(w)\right)$ of continuous linear mapping from $B_{\alpha,\beta}(w)$ into itself. The space $\mathcal{E}_{\alpha,\beta}(w)'$ dual of $\mathcal{E}_{\alpha,\beta}(w)$ is characterized as the subspace of $B_{\alpha,\beta}(w)'$ defining Hankel type convolution operators on $B_{\alpha,\beta}(w)$.

Throughout this paper we always denote by C a suitable positive constant that can change from one line to another one.

The space $\mathcal{H}_{\alpha,\beta}(w)$

In the sequel w is a function in $\mathcal{M}$. We now introduce the function spaces $\mathcal{H}_{\alpha,\beta}(w)$. A function $\phi \in L_{1,\alpha,\beta}$ is in $\mathcal{H}_{\alpha,\beta}(w)$ when ϕ and $h_{\alpha,\beta}(\phi)$ are smooth functions and, for every $m, n \in \mathbb{N}$,

$$u_{m,n}(\phi) = \underset{x \in (0,\infty)}{Sup}\, e^{m\,w(x)} \left| \left(\frac{1}{x} D\right)^n \phi(x) \right| < \infty,$$

and

$$v_{m,n}^{\alpha,\beta}(\phi) = \underset{x \in (0,\infty)}{Sup}\, e^{m\,w(x)} \left| \left(\frac{1}{x} D\right)^n h_{\alpha,\beta}(\phi)(x) \right| < \infty.$$

On $\mathcal{H}_{\alpha,\beta}(w)$ we consider the topology generated by the family

$$\left\{u_{m,n}, v_{m,n}^{\alpha,\beta}\right\}_{m,n \in \mathbb{N}}$$

of semi-norms.

In the following we establish some properties of $\mathcal{H}_{\alpha,\beta}(w)$ that can be proved by invoking well-known properties of the Hankel type transformation $h_{\alpha,\beta}$ and the conditions imposed on the function w.

Proposition 2.1 : (i) The space $\mathcal{H}_{\alpha,\beta}\,(w)$ is a Frechet space and it is continuously contained in $\mathcal{H}_{-1/2}$. Moreover if $w\,(x) = \log\,(1+x),\ x \in [0,\infty)$, then $\mathcal{H}_{\alpha,\beta}\,(w) = \mathcal{H}_{-1/2}$, where the equality is algebraical and topological.

(ii) The Hankel type transformation $h_{\alpha,\beta}$ is an automorphism of $\mathcal{H}_{\alpha,\beta}\,(w)$,

(iii) The Bessel type operator $\Delta_{\alpha,\beta} = x^{4\beta-2}\,D\,x^{4\alpha}\,D$ defines a continuous linear mapping from $\mathcal{H}_{\alpha,\beta}\,(w)$ into itself.

(iv) If P is a polynomial, then the mapping $\phi \to P(x^2)\,\phi$ is linear and continuous from $\mathcal{H}_{\alpha,\beta}\,(w)$ into itself.

We now introduce a new family of seminorms on $\mathcal{H}_{\alpha,\beta}\,(w)$ that is equivalent to $\left\{u_{m,n},\ v_{m,n}^{\alpha,\beta}\right\}_{m,n\,\in\,\mathbb{N}}$ and that will be very useful in the sequel.

Proposition 2.2: For every $m, n \in \mathbb{N}$, we define

$$A_{m,n}^{\alpha,\beta}\,(\phi) = \operatorname*{Sup}_{x\,\in\,(0,\infty)} e^{mw(x)}\left|\Delta_{\alpha,\beta}^{n}\,\phi\,(x)\right|,\ \ \phi \in \mathcal{H}_{\alpha,\beta}\,(w),$$

and

$$B_{m,n}^{\alpha,\beta}\,(\phi) = \operatorname*{Sup}_{x\,\in\,(0,\infty)} e^{mw(x)}\left|\Delta_{\alpha,\beta}^{n}\,h_{\alpha,\beta}\,(\phi)\,(x)\right|, \qquad \phi \in \mathcal{H}_{\alpha,\beta}\,(\phi),$$

where $\Delta_{\alpha,\beta}$ represents the Bessel type operator $x^{4\beta-2}\,D\,x^{4\alpha}\,D$.

The family $\left\{A_{m,n}^{\alpha,\beta}\ ,\ B_{m,n}^{\alpha,\beta}\right\}_{m,n\,\in\,\mathbb{N}}$ of semi-norms generates the topology of $\mathcal{H}_{\alpha,\beta}\,(w)$.

Proof: Proposition 2.1 (ii) and (iii) imply that the topology defined on $\mathcal{H}_{\alpha,\beta}\,(w)$ by $\left\{u_{m,n},\ v_{m,n}^{\alpha,\beta}\right\}_{m,n\,\in\,\mathbb{N}}$ is stronger than the one induced on it by $\left\{A_{m,n}^{\alpha,\beta}\ ,\ B_{m,n}^{\alpha,\beta}\right\}_{m,n\,\in\,\mathbb{N}}$.

Now we will see that $\left\{A_{m,n}^{\alpha,\beta},\ B_{m,n}^{\alpha,\beta}\right\}_{m,n\,\in\,\mathbb{N}}$ generates on $\mathcal{H}_{\alpha,\beta}\,(w)$ a topology finer than the one defined on it by $\left\{u_{m,n},\ v_{m,n}^{\alpha,\beta}\right\}_{m,n\,\in\,\mathbb{N}}$.

For every $k \in \mathbb{N}$ and $\phi \in \mathcal{H}_{\alpha,\beta}\,(w)$, we have that

$$\left(\frac{1}{x}\,D\right)^{k}\phi\,(x) = x^{-2(\alpha-\beta)-2k}\int_0^{x} x_k \int_0^{x_k} x_{k-1} \ldots\ldots \int_0^{x_2} x_1^{4\alpha}\,\Delta_{\alpha,\beta}^{k}\,\phi\,(x_1)\,dx_1 \ldots .\,dx_k,\ \ x \in (0,\infty), \qquad (2.1)$$

and

$\left(\frac{1}{x}\,D\right)^k \phi\,(x) =$

$(-1)^k x^{-2(\alpha-\beta)-2k}\ \int_x^\infty x_k \int_{x_k}^\infty x_{k-1} \dots \int_{x_2}^\infty x_1^{4\alpha}\, \Delta_{\alpha,\beta}^k\, \phi\,(x_1)\, dx_1 \dots dx_k, x\ \in (0,\infty).$ (2.2)

To prove (2.1) and (2.2), we must proceed inductively. We will show that (2.1). To see (2.2), we can argue in a similar way.

Formula (2.1) holds when $k = 1$. Infact, according to Proposition 2.1 (i) and by [1, Lemma 8 b], it has, for every $\phi\ \in \mathcal{H}_{\alpha,\beta}\,(w)$

$$h_{3\alpha,\beta}\left(\left(\frac{1}{x}\,D\right)\phi\right) = -h_{\alpha,\beta}\,(\phi). \tag{2.3}$$

Moreover, by partial integration and by [20(7), Chapter 5], since the function $z^{\alpha+\beta}\, J_{\alpha-\beta}\,(z)$ is bounded on $(0,\infty)$, it has, for every $y\ \in\ (0,\infty)$ and $\phi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$,

$$h_{3\alpha,\beta}\left(x^{-6\alpha-2\beta}\ \int_0^x x_1^{4\alpha}\, \Delta_{\alpha,\beta}\, \phi\ (x_1)\, dx_1\right)(y) \tag{2.4}$$

$$= -y^{-2} \int_0^\infty \frac{d}{dx}\left((xy)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(xy)\right) \int_0^x x_1^{4\alpha}\, \Delta_{\alpha,\beta}\, \phi\,(x_1)\, dx_1\, dx$$

$$= y^{-2}\, h_{\alpha,\beta}\left(\Delta_{\alpha,\beta}\, \phi\right)(y)$$

$$= -h_{\alpha,\beta}\,(\phi)\,(y).$$

From (2.3) and (2.4) we deduce that (2.1) is true for every $\phi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$ when $k = 1$.

We now suppose that $l\ \in\ \mathbb{N}$ and that, for every $\phi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$,

we have

$$\left(\frac{1}{x}\,D\right)^l\ \phi\,(x) = x^{-2(\alpha-\beta)-2l}\ \int_0^x x_l\ \int_0^{xl} x_{l-1} \dots \int_0^{x_2} x_1^{4\alpha}\, \Delta_{\alpha,\beta}^l\, \phi\,(x_1)\, dx_l \dots dx_1,$$

$x\ \in\ (0,\infty).$ (2.5)

We have to see that (2.5) holds when l is replaced by $l+1$ for every $\phi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$. Let $\phi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$. According to [1, Lemma 8], we can write

$$\left(\frac{1}{x}\,D\right)^{l+1} \phi = (-1)^{l+1}\, h_{\alpha-\beta+l+1}\,(h_{\alpha,\beta}\,\phi)\,.$$

On the other hand, it is easy to see that the induction hypothesis (2.5) it deduces that, since $\Delta_{\alpha,\beta}\,\phi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$, Proposition 2.1,

$$x^{-2(\alpha-\beta)-2(l+1)} \int_0^x x_{l+1} \int_0^{x_{l+1}} x_l \dots \int_0^{x_2} x_1^{4\alpha}\, \Delta_{\alpha,\beta}^{l+1}\, \phi\,(x_1)\ dx_1 \dots dx_{l+1}$$

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$$= \Lambda_{\alpha,\beta,l} \left(\left(\frac{1}{x} D \right)^{l} \Delta_{\alpha,\beta}\, \phi \right) (x)\,,\ x \in (0, \infty) \qquad (2.6)$$

where $\Lambda_{\alpha,\beta}$ denotes the operator defined by

$\left(\Lambda_{\alpha,\beta}\, \psi\right)(x) = x^{-6\alpha-2\beta} \int_0^x t^{2(\alpha-\beta)+l}\, \psi(t)\, dt,\ \ x \in (0,\infty)$, for every $\psi \in \mathcal{H}_{\alpha,\beta}(w)$.

Moreover, from (2.3), it follows that

$$\left(\frac{1}{x} D\right)^{l} \Delta_{\alpha,\beta}\, \phi = \Delta_{\alpha,\beta,l} \left(\frac{1}{x} D\right)^{l} \phi\,. \qquad (2.7)$$

On the other hand, by partial integration and by [1, Lemma 8b] we obtain that, for every $\psi \in \mathcal{H}_{-1/2}$,

$h_{\alpha,\beta,l+1}\left(\Lambda_{\alpha,\beta,l}\, \Delta_{\alpha,\beta,l}\, \psi\right)(y)$

$$= -y^{-2} \int_0^{\infty} \frac{d}{dx} \left((xy)^{-\alpha+\beta-l} J_{\alpha-\beta+l}(xy) \right) \int_0^{x} t^{2(\alpha-\beta)+2l+1} \Delta_{\alpha,\beta,l}\, \psi(t)\, dt\, dx$$

$$= -h_{\alpha,\beta,l}(\psi)(y)\,, \qquad y \in (0,\infty).$$

Hence

$$\Lambda_{\alpha,\beta,l}\, \Delta_{\alpha,\beta,l}\ \psi = \left(\frac{1}{x} D\right) \psi\,,\ \psi \in \mathcal{H}_{-1/2}\,. \qquad (2.8)$$

From (2.6), (2.7) and (2.8), according to proposition 2.1 (i), it implies that

$\left(\frac{1}{x} D\right)^{l+1} \phi(x) =$

$x^{-2(\alpha-\beta)-2(l+1)} \int_0^x x_{l+1} \int_0^{x_{l+1}} x \ldots\ldots \int_0^{x_2} x_1^{4\alpha}\, \Delta_{\alpha,\beta}^{l+1}\, \phi(x_1)\, dx_1 \ldots\ldots dx_{l+1},\ \ x \in (0,\infty).$

Thus (2.1) is proved.

Now let $m, n \in \mathbb{N}$. Assume that $\phi \in \mathcal{H}_{\alpha,\beta}(w)$. From (2.1) it follows that

$$e^{mw(x)} \left| \left(\frac{1}{x} D\right)^{n} \phi(x) \right|$$

$$\leq C \sup_{Z \in (0,\infty)} \left|\Delta_{\alpha,\beta}^{n}\, \phi(z)\right| x^{-2(\alpha-\beta)-2n} \int_0^x x_n \int_0^{x_n} x_{n-1} \ldots \ldots \int_0^{x_2} x_1^{4\alpha}\, dx_1 \ldots\ldots dx_n$$

$$\leq C \sup_{Z \in (0,\infty)} \left|\Delta_{\alpha,\beta}^{n}\, \phi(z)\right|\,, \qquad x \in (0,1).$$

Also, by using (2.2), since w is increasing and it satisfies the (iii) property, we obtain for $l \in \mathbb{N}$ large enough,

$$e^{mw(x)} \left| \left(\frac{1}{x} D\right)^n \phi(x) \right| \leq x^{-2(\alpha-\beta)-2n} \int_x^\infty x_n \int_{x_n}^\infty x_{n+1} \dots \int_{x_2}^\infty x_1^{4\alpha} e^{mw(x_1)} \left|\Delta_{\alpha,\beta}^n \phi(x_1)\right| \times dx_1 \dots dx_n$$

$$\leq C \sup_{z \in (0,\infty)} e^{(m+l)w(z)} \left|\Delta_{\alpha,\beta}^n \phi(z)\right|, \quad x \geq 1.$$

Hence, it concludes that, for a certain $l \in \mathbb{N}$,

$$u_{m,n}(\phi) \leq C\, A_{m+l,n}^{\alpha,\beta}(\phi).$$

According to Proposition 2.1 (ii) $h_{\alpha,\beta}(\phi)$ is also in $\mathcal{H}_{\alpha,\beta}(w)$ and then the following inequality also holds

$$v_{m,n}^{\alpha,\beta}(\phi) \leq C\, B_{m+l,n}^{\alpha,\beta}(\phi).$$

Thus we prove that the topology generated by $\left\{A_{m,n}^{\alpha,\beta}, B_{m,n}^{\alpha,\beta}\right\}_{m,n \in \mathbb{N}}$ on $\mathcal{H}_{\alpha,\beta}(w)$ is finer than the one induced on it by $\left\{u_{m,n},\ v_{m,n}^{\alpha,\beta}\right\}_{m,n \in \mathbb{N}}$ and thus the proof is completed.

Through the proof of Proposition 2.2 we also show the following characterizations of the space $\mathcal{H}_{\alpha,\beta}(w)$.

Proposition 2.3: A function $\phi \in \mathcal{H}_{\alpha,\beta}(w)$ if and only if $\phi \in \mathcal{H}_{-1/2}$ and ϕ satisfies one of the following three conditions:

(i) For every $m, n \in \mathbb{N}$, $A_{m,n}^{\alpha,\beta}(\phi) < \infty$ *and* $B_{m,n}^{\alpha,\beta}(\phi) < \infty$,

(ii) For every $m, n \in \mathbb{N}$, $A_{m,n}^{\alpha,\beta}(\phi) < \infty$ *and* $v_{m,n}^{\alpha,\beta}(\phi) < \infty$,

(iii) For every $m, n \in \mathbb{N}$, $u_{m,n}(\phi) < \infty$ *and* $B_{m,n}^{\alpha,\beta}(\phi) < \infty$.

Moreover, the families of seminorms $\left\{A_{m,n}^{\alpha,\beta}, B_{m,n}^{\alpha,\beta}\right\}_{m,n \in \mathbb{N}}$, $\left\{A_{m,n}^{\alpha,\beta}, v_{m,n}^{\alpha,\beta}\right\}_{m,n \in \mathbb{N}}$ and $\left\{u_{m,n},\ B_{m,n}^{\alpha,\beta}\right\}_{m,n \in \mathbb{N}}$ generates the topology of $\mathcal{H}_{\alpha,\beta}(w)$.

We now analyze the behavior of Hankel type translation operator on $\mathcal{H}_{\alpha,\beta}(w)$.

Proposition 2.4: (i) Let $x \in (0,\infty)$. The Hankel type translation operator τ_x defines a continuous linear mapping from $\mathcal{H}_{\alpha,\beta}(w)$ into itself.

(ii) Let $\phi \in \mathcal{H}_{\alpha,\beta}(w)$. The (nonlinear) mapping F_ϕ defined by $F_\phi(x) = \tau_x \phi$, $x \in (0,\infty)$ is continuous from $[0,\infty)$ into $\mathcal{H}_{\alpha,\beta}(w)$.

Proof: (i) Let $\phi \in \mathcal{H}_{\alpha,\beta}(w)$ and $m, n \in \mathbb{N}$. Since $\Delta_{\alpha,\beta}\,\tau_x\,\phi = \tau_x\,\Delta_{\alpha,\beta}\,\phi$ ([12, Proposition 2.1]) and since w is increasing and it satisfies the property (i), we can write

$$e^{mw(y)}\left|\Delta^n_{\alpha,\beta}(\tau_x\,\phi)(y)\right|$$

$$\leq e^{mw(y)}\,\tau_x\left(\left|\Delta^n_{\alpha,\beta}\,\phi\right|\right)(y)$$

$$\leq e^{m(w(y)-w(|x-y|))}\int_{|x-y|}^{x+y} D(x,y,z)\,e^{mw(z)}\left|\Delta^n_{\alpha,\beta}\,\phi(z)\right|\frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dz$$

$$\leq e^{mw(x)}\sup_{z\,\in\,(0,\infty)} e^{mw(z)}\left|\Delta^n_{\alpha,\beta}\,\phi(z)\right|\int_0^\infty D(x,y,z)\frac{z^{4\alpha}}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\,dz,$$

for each $y \in (0,\infty)$.

Hence by [11, (2)], it concludes

$$A^{\alpha,\beta}_{m,n}(\tau_x\,\phi) \leq e^{mw(x)}\,A^{\alpha,\beta}_{m,n}(\phi). \tag{2.9}$$

On the other hand, by [3,(3.1)] and [20, (7), Chapter 5], since the function $z^{-(\alpha-\beta)}J_{\alpha-\beta\,(z)}$ is bounded on $(0,\infty)$, it follows

$$e^{mw(y)}\left|\left(\frac{1}{y}D\right)^m h_{\alpha,\beta}(\tau_x\phi)(y)\right|$$

$$= e^{mw(y)}\left|\left(\frac{1}{y}D\right)^n\left(2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)(xy)^{-(\alpha-\beta)}\,J_{\alpha-\beta}(xy)\,h_{\alpha,\beta}(\phi)(y)\right)\right|$$

$$\leq C\sum_{j=0}^{n} e^{mw(y)}\left|\left(\frac{1}{y}D\right)^{n-j} h_{\alpha,\beta}(\phi)(y)\right| x^{2j},\quad y \in (0,\infty).$$

Then

$$v^{\alpha,\beta}_{m,n}(\tau_x\,\phi) \leq C\,(1+x^{2n})\,\textstyle\sum_{j=0}^{n} v^{\alpha,\beta}_{m,j}(\phi)\ . \tag{2.10}$$

From (2.9) and (2.10) we deduce that τ_x is continuous from $\mathcal{H}_{\alpha,\beta}(w)$ into itself.

(ii) Let $\phi \in \mathcal{H}_{\alpha,\beta}(w)$. Assume that $x_0 \in (0,\infty)$ and $m, n \in \mathbb{N}$. We can write for every $x \in [(x_0/2),\ (3x_0/2)]$ and $y \geq 2x_0$,

$$e^{mw(y)}\left|\Delta^n_{\alpha,\beta}\left((\tau_x\phi)-(\tau_{x_0}\phi)\right)(y)\right|$$

$$\leq e^{(m+1)[w(y)-w(y-(3x_0/2))]-w(y)}\sup_{z\,\in\,(0,\infty)} e^{(m+1)w(z)}\left|\Delta^n_{\alpha,\beta}\,\phi(z)\right|$$

$$\times\int_{y-(3x_0/2)}^{y+(3x_0/2)}\left|D(x,y,z)-D(x_0,y,z)\right|\frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dz$$

$$\leq 2\, e^{(m+1)w(3x_0/2)-w(y)} \underset{z\,\in\,(0,\infty)}{Sup}\; e^{(m+1)w(z)} \left|\Delta^n_{\alpha,\beta}\, \phi(z)\right| .$$

Hence, if $\epsilon > 0$, then there exists $y_1 \geq 2x_0$ such that, for every $x \in [(x_0/2)\,,(3x_0/2)]$ and $y \geq y_1$,

$$e^{mw(y)} \left|\Delta^n_{\alpha,\beta}\left((\tau_x\phi) - (\tau_{x_0}\phi)\right)(y)\right| < \epsilon.$$

On the other hand, since w is increasing on $[0,\infty)$, it has

$$\underset{y\,\epsilon\,(0,\; y_1)}{Sup}\; e^{mw(y)} \left|\Delta^n_{\alpha,\beta}\left((\tau_x\phi) - (\tau_{x_0}\phi)\right)(y)\right|$$

$$\leq e^{mw(y_1)} \underset{y\,\in\,(0,y_1)}{Sup} \left|\Delta^n_{\alpha,\beta}\left((\tau_x\phi) - (\tau_{x_0}\phi)\right)(y)\right|.$$

Therefore, according to [12, p.359], since $\Delta_{\alpha,\beta}$ is a continuous operator from $\mathcal{H}_{-1/2}$ into itself, we deduce that if $\epsilon > 0$, then

$$\underset{y\,\in\,(0,y_1)}{Sup}\; e^{mw(y)} \left|\Delta^n_{\alpha,\beta}\left((\tau_x\phi) - (\tau_{x_0}\phi)\right)(y)\right| < \epsilon,$$

provided that $x \in (0,\infty)$ and $|x - x_0| < \delta$, for some $\delta > 0$.

Thus we conclude that, for every $\epsilon > 0$, there exists $\delta > 0$ for which

$$A^{\alpha,\beta}_{m,n}\left(\tau_x\phi - \tau_{x_0}\phi\right) < \epsilon\,,$$

when $x \in (0,\infty)$ and $|x - x_0| < \delta$.

Moreover, the Leibniz rule and again [3,(3.1)] and [20(7), Chapter 5] lead to

$$\left(\frac{1}{y}\frac{d}{dy}\right)^n \left(h_{\alpha,\beta}\,(\tau_x\phi) - \tau_{x_0}\phi\,(y)\right)$$

$$= 2^{\alpha-\beta}\Gamma(3\alpha+\beta)\sum_{j=0}^{n}\binom{n}{j}(-1)^j\left(\frac{1}{y}\frac{d}{dy}\right)^{n-j} h_{\alpha,\beta}\,(\phi)\,(y)$$

$$\times\left(x^{2j}\,(xy)^{-\alpha+\beta-j}\,J_{\alpha-\beta+j}\,(xy) - x_0^{2j}\,(x_0y)^{-\alpha+\beta-j}J_{\alpha-\beta+j}\,(x_0y)\right), \qquad x,y \in (0,\infty).$$

Hence, the boundedness of the function $z^{-(\alpha-\beta)}J_{\alpha-\beta}\,(z)$, $z \in (0,\infty)$, implies that if $\epsilon > 0$,

$$e^{mw(y)}\left|\left(\frac{1}{y}\frac{d}{dy}\right)^n\left(h_{\alpha,\beta}\left(\tau_x\phi - \tau_{x_0}\phi\right)(y)\right)\right|$$

$$\leq C\, e^{-w(y)}\sum_{j=0}^{n}\left(x^{2j} + x_0^{2j}\right) v^{\alpha,\beta}_{m+1,n-j}\,(\phi) < \epsilon,$$

for each $x \in (0,\; 2x_0)$ and $y \geq y_1$, where y_1 is a large enough positive number.

On the other hand, since the function $f_j\,(x,y) = 2^{2j}\,(xy)^{-\alpha+\beta-j}\,J_{\alpha-\beta+j}\,(xy)$, $x,y \in [0,\infty)$, is continuous (and hence uniformly continuous in each compact subset of $[0,\infty) \times [0,\infty)$), for every $j \in \mathbb{N}$, if $\epsilon > 0$ we can find $\delta > 0$ such that $|f_j\,(x,y) - f_j\,(x_0,y)| < \epsilon$, for every $y \in [0,y_1]$, $x \in [0,\infty)$, $|x - x_0| < \delta$ and $j = 0, \ldots\ldots, n$. Then

$$\underset{y\,\in\,(0,y_1)}{Sup}\; e^{mw(y)} \left|\left(\frac{1}{y}\frac{d}{dy}\right)^n \left(h_{\alpha,\beta}\,\tau_x\,\phi - \tau_{x_0}\phi\,(y)\right)\right| \le C\,\epsilon \sum_{j=0}^{n} u_{m,j}^{\alpha,\beta}\,(\phi),$$

for every $x \in (0,\infty)$ and $|x - x_0| < \delta$.

Thus, it is concluded that for every $\epsilon > 0$, there exists $\delta > 0$ such that

$$u_{m,n}^{\alpha,\beta}\left(\tau_x\phi - \tau_{x_0}\phi\right) < \epsilon\,,$$

provided that $x \in (0,\infty)$ and $|x - x_0| < \delta$.

Hence F_ϕ is a continuous function on x_0.

To see that F_ϕ is continuous in $x = 0$, we can proceed in a similar way.

Thus proof is completed.

Next, we study the pointwise multiplication and the Hankel type convolution on $\mathcal{H}_{\alpha,\beta}\,(w)$.

Proposition 2.5: The bilinear mappings defined by

$$(\phi,\psi) \to \phi\psi$$

and

$$(\phi,\psi) \to \phi\,\#\,\psi$$

are continuous from $\mathcal{H}_{\alpha,\beta}\,(w) \times \mathcal{H}_{\alpha,\beta}\,(w)$ into $\mathcal{H}_{\alpha,\beta}\,(w)$.

Proof: By virtue of the interchange formula [12, Theorem 2d]

$$h_{\alpha,\beta}\,(\phi\#\psi) = h_{\alpha,\beta}\,(\phi)\,h_{\alpha,\beta}\,(\psi), \qquad \phi,\psi \in \mathcal{H}_{\alpha,\beta}\,(w),$$

the continuity of the pointwise multiplication mapping is equivalent to the one of the Hankel type convolution mapping.

Let $m,n \in \mathbb{N}$. Assume that $\phi,\psi \in \mathcal{H}_{\alpha,\beta}\,(w)$, we can write, from the Leibniz rule, that

$$u_{m,n}\,(\phi\psi) \le C \sum_{j=0}^{n} u_{m,n,j}\,(\phi)\,u_{0,j}\,(\psi)\,.$$

On the other hand, since $\Delta_{\alpha,\beta}\,(\phi\#\psi) = \left(\Delta_{\alpha,\beta}\,\phi\right)\#\,\psi$ [14, Proposition 2.2] and since w is increasing on $[0,\infty)$ and it satisfies the property (i) of Section 1, it has

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$$e^{mw(x)}\left|\Delta^n_{\alpha,\beta}\, h_{\alpha,\beta}\,(\phi\psi)\,(x)\right|$$

$$= \; e^{mw(x)}\;\left|\left(\left(\Delta^n_{\alpha,\beta}\, h_{\alpha,\beta}(\phi)\right)\#\, h_{\alpha,\beta}\,(\psi)\right)(x)\right|$$

$$\leq \; e^{mw(x)}\int_0^{\infty}\left|\Delta^n_{\alpha,\beta}\left(h_{\alpha,\beta}\phi\right)(y)\right|\; e^{-mw(|x-y|)}$$

$$\times \int_{|x-y|}^{x+y} D(x,y,z)\left|h_{\alpha,\beta}\,(\psi)\,(z)\right| e^{mw(z)}\frac{z^{4\alpha}}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\,dz\;\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy$$

$$\leq \int_0^{\infty}\left|\Delta^n_{\alpha,\beta}\left(h_{\alpha,\beta}\,\phi\right)(y)\right| e^{mw(y)}\int_{|x-y|}^{x+y} D(x,y,z)\left|h_{\alpha,\beta}\,(\psi)(z)\right| e^{mw(z)}$$

$$\times\;\frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dz\,.\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy\;, x\;\in(0,\infty).$$

Hence, since w verifies the property (i) of Section 1 and by taking into account [11], we can conclude

$$B^{\alpha,\beta}_{m,n}\,(\phi\psi)\;\leq C\;B^{\alpha,\beta}_{m+l,n}\,(\phi)\;B^{\alpha,\beta}_{m,0}\,(\psi)\;, \text{for some } l\;\in\;\mathbb{N}.$$

By virtue of Proposition 2.3, we have proved that the pointwise multiplication defines a continuous mapping from

$$\mathcal{H}_{\alpha,\beta}(w)\times\mathcal{H}_{\alpha,\beta}(w)\;into\;\mathcal{H}_{\alpha,\beta}(w).$$

Thus the proof is completed.

Remark 1: The last proposition shows that each function in $\mathcal{H}_{\alpha,\beta}\,(w)$ defines a multiplier in $\mathcal{H}_{\alpha,\beta}\,(w)$. Also, in the proof of Proposition 2.4, it was established that for every $x\;\in\;(0,\infty)$ the function f_x defined by

$$f_x(y) = \;(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}\,(xy)\,, \qquad y\;\in\;(0,\infty),$$

is a multiplier of $\mathcal{H}_{\alpha,\beta}\,(w)$.

In [2] we introduced the space $B_{\alpha,\beta}\,(w)$ (see Section 1 for definitions). $B_{\alpha,\beta}\,(w)$ can be considered as a Beurling type function space for the Hankel $h_{\alpha,\beta}$ transformation. In the following we establish that $B_{\alpha,\beta}\,(w)$ is dense subset of $\mathcal{H}_{\alpha,\beta}\,(w)$.

Proposition 2.6: The space $B_{\alpha,\beta}\,(w)$ is continuously contained in $\mathcal{H}_{\alpha,\beta}\,(w)$. Moreover, $B_{\alpha,\beta}\,(w)$ is a dense subspace of $\mathcal{H}_{\alpha,\beta}\,(w)$.

Proof: Let $\phi \in B^{a}_{\alpha,\beta}(w)$, where $a > 0$. Since ϕ and $h_{\alpha,\beta}(\phi) \in L_{\alpha,\beta,1}$, according to [11, Corollary 2], it has

$$\phi(x) = \int_0^\infty (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, h_{\alpha,\beta}(\phi)(y)\, y^{4\alpha}\, dy, \qquad x \in (0,\infty).$$

Hence by invoking [20 (7), Chapter 5], since $z^{-(\alpha-\beta)} J_{\alpha-\beta}(z)$ is a bounded function on $(0,\infty)$ and w satisfies the property (iii) of Section 1 for every $m,n \in \mathbb{N}$, we can find $l \in \mathbb{N}$ for which

$$u_{m,n}(\phi) \leq C\ Sup_{x \in (0,a)}\, e^{mw(x)} \int_0^\infty y^{2n+4\alpha} \left|h_{\alpha,\beta}(\phi)(y)\right| dy \leq C\, \delta_l^{\alpha,\beta}(\phi). \qquad (2.11)$$

Here C is a positive constant that is not dependent on ϕ.

By virtue of the Paley-Wiener type theorem for the Hankel type transform on $B^{a}_{\alpha,\beta}(w)$ ([2, Proposition 2.6]), $h_{\alpha,\beta}(\phi)$ is an even entire function and for every $m \in \mathbb{N}$, there exists $C_m > 0$ for which

$$\left|h_{\alpha,\beta}(\phi)(x+iy)\right| \leq C_m\, e^{-mw(x)+(a+1)|y|},\ x,y \in \mathbb{R}. \qquad (2.12)$$

According to the well-known Cauchy integral formula, we can write

$$\frac{d^l}{dx^l} h_{\alpha,\beta}(\phi)(x) = \frac{l!}{2\pi i} \int_{C_x} \frac{h_{\alpha,\beta}(\phi)(z)}{(z-x)^{l+1}}\, dz,\ \ l \in \mathbb{N}\ and\ x \in \mathbb{R}, \qquad (2.13)$$

where C_x represents the circled path having bi-parametric representation $z = x + e^{i\theta}$, $\theta \in [0,2\pi)$.

Let $m,n \in \mathbb{N}$. From (2.12) and (2.13), it follows, since w satisfies the property (i), that

$$\left|\frac{d^n}{dx^n} h_{\alpha,\beta}(\phi)(x)\right| \leq C \int_0^{2\pi} e^{-mw(x+\cos\theta)+(a+1)|\sin\theta|} d\theta \leq C\, e^{-mw(x)},\ \ x \geq 1.$$

Thus, it follows

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n h_{\alpha,\beta}(\phi)(x)\right| \leq C\, e^{-mw(x)}, \qquad x \geq 1.$$

Moreover, by using again the above-mentioned properties of the Bessel type functions, we have

$$\left|\left(\frac{1}{x}\frac{d}{dx}\right)^n h_{\alpha,\beta}(\phi)(x)\right| \leq C \int_0^a y^{2n+4\alpha} |\phi(y)|\, dy \leq C\, u_{0,0}(\phi),\ \ x \in (0,1).$$

Thus we conclude that $v^{\alpha,\beta}_{m,n}(\phi) < \infty$.

We have proved that $B_{\alpha,\beta}^{a}(w) \subset \mathcal{H}_{\alpha,\beta}(w)$.

To see that $B_{\alpha,\beta}^{a}(w)$ is continuously contained in $\mathcal{H}_{\alpha,\beta}(w)$ we will use the closed graph theorem. Assume that $\{\phi_v\}_{v\in\mathbb{N}}$ is a sequence in $B_{\alpha,\beta}^{a}(w)$ such that $\phi_v \to \phi$ as $v \to \infty$, in $B_{\alpha,\beta}^{a}(w)$ and $\phi_v \to \psi$ as $v \to \infty$, in $\mathcal{H}_{\alpha,\beta}(w)$. It is clear that $\phi_v(x) \to \psi(x)$ *as* $v \to \infty$ for every $x \in (0,\infty)$. Moreover, from (2.11) we deduce that $\phi_v(x) \to \phi(x)$ as $v \to \infty$ for each $x \in (0,\infty)$. Hence $\phi = \psi$. Thus we show that $B_{\alpha,\beta}^{a}(w) \subset \mathcal{H}_{\alpha,\beta}(w)$ is continuous.

We now see that $v_{\alpha,\beta}(w)$ is a dense subset of $\mathcal{H}_{\alpha,\beta}(w)$. According to [2, Proposition 2.18] we choose $\psi \in B_{\alpha,\beta}^{2}(w)$ such that $0 \le \psi \le 1$ and $\psi(x) = 1, x \in (0,1)$. Assume that $\phi \in \mathcal{H}_{\alpha,\beta}(w)$. We define for every $l \in \mathbb{N} - \{0\}$, $\psi_l(x) = \psi(x/l)$, $x \in (0,\infty)$ and $\phi_l = \psi_l\,\phi$. Let $m,n \in \mathbb{N}$. The Leibniz rule leads to, for every $l \in \mathbb{N} - \{0\}$,

$$e^{mw(x)}\left|\left(\frac{1}{x}D\right)^{n}\left(\phi_l(x) - \phi(x)\right)\right| \le S_l^1(x) + S_l^2(x), \qquad x \in (0,\infty),$$

where

$$S_l^1(x) = \sum_{j=0}^{n-1}\binom{n}{j} e^{mw(x)}\left|\left(\frac{1}{x}D\right)^{j}\phi(x)\right|\left|\left(\frac{1}{x}D\right)^{n-j}\psi\left(\frac{x}{l}\right)\right|, \qquad x \in (0,\infty),$$

and

$$S_l^2(x) = e^{mw(x)}\left|\left(\frac{1}{x}D\right)^{l}\phi(x)\right|\left|\psi\left(\frac{x}{l}\right) - 1\right|, \qquad x \in (0,\infty).$$

Standard arguments allow us now to conclude that

$$u_{m,n}(\phi_l - \phi) \to 0, \qquad as\ l \to \infty.$$

On the other hand, by [11, Theorem 2d], since $\psi_l(0) = 1$, $l \in \mathrm{N} - \{0\}$, we can write

$\Delta_{\alpha,\beta}^{n}\,h_{\alpha,\beta}(\phi_l - \phi)(x)$.

$$= \left(h_{\alpha,\beta}(\psi_l)\,\#\,\Delta_{\alpha,\beta}^{n}\,h_{\alpha,\beta}(\phi)\right)(x) - \Delta_{\alpha,\beta}^{n}\,h_{\alpha,\beta}(\phi)(x)$$

$$= \int_0^\infty h_{\alpha,\beta}(\psi_l)(y)\left(\tau_x\left(\Delta_{\alpha,\beta}^{n}\,h_{\alpha,\beta}(\phi)\right)(y) - \Delta_{\alpha,\beta}^{n}\,h_{\alpha,\beta}(\phi)(x)\right)\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy,$$

for each $x \in (0,\infty)$ and $l \in N - \{0\}$.

Fix $l \in N-\{0\}$. To simplify we denote by $\Phi = \Delta^n_{\alpha,\beta}\, h_{\alpha,\beta}\,(\phi)$. It is not hard to see that $h_{\alpha,\beta}\,(\psi_l)\,(y) = l^{2(3\alpha+\beta)}\, h_{\alpha,\beta}\,(\psi)(yl),\ y \in (0,\infty)$. Then

$$\Delta^n_{\alpha,\beta}\, h_{\alpha,\beta}\,(\phi_l - \phi)\,(x)$$

$$= \int_0^\infty h_{\alpha,\beta}\,(\psi)\,(y)\left(\tau_x\,(\Phi)\left(\frac{y}{l}\right) - \Phi(x)\right)\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy, \qquad x \in (0,\infty).$$

We now consider $d \in (0,1)$ that will be specified later. We divide the last integral into two parts.

According to [11 (2)], since w is an increasing function on $[0,\infty)$, we have that

$$\left|\int_{x+l^d} h_{\alpha,\beta}(\psi)(y)\int_{|x-y/l|}^{x+y/l} D\left(x,\frac{y}{l},z\right)\big(\Phi(z) - \Phi(x)\big)\frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dz\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy\right|$$

$$\leq C \sup_{z\,\in\,(0,\infty)} |\Phi\,(z)| \int_{x+l^d}^\infty \left|h_{\alpha,\beta}\,(\psi)\,(y)\right| y^{4\alpha}\,dy$$

$$\leq C \int_{x+l^d}^\infty e^{-(m+k)w(y)}\, y^{4\alpha}\,dy \,.\sup_{z\,\in\,(0,\infty)} |\Phi(z)| \sup_{z\,\in\,(0,\infty)} \left|h_{\alpha,\beta}\,(\psi)\,(z)\right| e^{(m+k)\,w\,(z)}$$

$$\leq C\, e^{-mw(z)} \int_{l^d}^\infty e^{-kw(y)}\, y^{4\alpha}\,dy \,.\sup_{z\,\in\,(0,\infty)} |\Phi(z)| \sup_{z\,\in\,(0,\infty)} \left|h_{\alpha,\beta}\,(\psi)\,(z)\right| e^{(m+k)\,w\,(z)},$$

for every $x \in (0,\infty)$ and $k \in \mathbb{N}$.

Hence, since w satisfies the property (i) of Section 1, by choosing $k \in \mathbb{N}$ large enough it follows that

$$\sup_{x\,\in(0,\infty)}\left|e^{mw(x)}\int_{x+l^d}^\infty h_{\alpha,\beta}\,(\psi)\,(y)\int_{|x-y/l|}^{x+y/l} D\,(x,y/l,z)\,\big(\Phi(z)-\Phi(x)\big) \times \frac{z^{4\alpha}}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\,dz\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha)}\,dy\right|$$

$\leq C \int_{l^d}^{\infty} e^{-kw(y)}\, y^{4\alpha}\, dy\, Sup_{z \in (0,\infty)} |\Phi(z)|\, Sup_{z \in (0,\infty)} \left| h_{\alpha,\beta}\,(\psi)\,(z)\right|\, e^{(m+k)\,w\,(z)} \to 0,$

$as\; l \to \infty$

On the other hand, by again using [11, (2)], one obtains, for every $x \in (0,\infty)$,

$$\left| e^{mw(x)} \int_{0}^{x+l^d} h_{\alpha,\beta}\,(\psi)\,(y) \int_{|x-y/l|}^{x+y/l} D(x,y/l,z)\,\big(\Phi(z)-\Phi(x)\big) \times \frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\, dz\, \frac{y^{4\alpha}}{2^{\alpha-\beta}\,\Gamma\,(3\alpha+\beta)}\, dy \right|$$

$$\leq C \underset{z \in (0,\infty)}{Sup} \left| h_{\alpha,\beta}\,(\phi)(z)\right| e^{mw(x)} (x+l^d)^{6\alpha+2\beta} \underset{\substack{|x-y/\iota| \leq z \leq x+y/\iota \\ 0<y<x+l^d}}{\text{Sup}} |\Phi(z)-\Phi(x)|,$$

Moreover, we have that, for each $\eta \in (0, x+l^d)$ and $x \in (0,\infty)$,

$$\left|\Phi\left(x+\frac{\eta}{l}\right)-\Phi\,(x)\right| \leq \int_{x}^{x+(n/l)} \left|\frac{d}{dt}\,\Phi(\mathrm{t})\right| dt$$

$$\leq \frac{1}{l}\,(x+l^d) \underset{-x-l^d \leq \xi \leq x+l^d}{Sup} \left|\left(\frac{d}{dt}\,\Phi\right)\left(x+\frac{\xi}{l}\right)\right|.$$

Also, we can write

$$\left|\Phi\left(x+\frac{\eta}{l}\right)-\Phi\,(x)\right| \leq \frac{1}{l}\,(x+l^d) \underset{-x-l^d \leq \xi \leq x+l^d}{Sup} \left|\left(\frac{d}{dt}\,\Phi\right)\left(x+\frac{\xi}{l}\right)\right|,$$

for each $x \in (0,\infty)$ and $\eta \in (-x-l^d, 0)$.

If it is necessary above we consider the even and smooth extension of Φ to $\mathbb{R}$. Hence, it has

$$\left| e^{mw(x)} \int_{0}^{x+l^d} h_{\alpha,\beta}\,(\psi)\,(y) \int_{|x-y/l|}^{x+y/l} D(x,y/l,z)\,\big(\Phi\,(z)-\Phi(x)\big) \times \frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\, dz\, \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\, dy \right|$$

$$\leq C \underset{z \in (0,\infty)}{Sup} \left| h_{\alpha,\beta}\,(\psi)(z)\right| e^{mw(x)} \frac{1}{l}\,(x+l^d)^{10\alpha+6\beta} \underset{-x-l^d \leq \xi \leq x+l^d}{Sup} \left|\left(\frac{1}{t}\,\frac{d}{dt}\,\Phi\right)\left(x+\frac{\xi}{l}\right)\right|$$

$$\leq C \sup_{z\in(0,\infty)} \left|h_{\alpha,\beta}\,(\psi)(z)\right| e^{mw(x)-kw\left(x-\left(\frac{x}{l}\right)-l^{d-1}\right)} \frac{1}{l}(x+l^d)^{10\alpha+6\beta}$$

$$\times \sup_{z\in(0,\infty)} \left|\frac{1}{z}\frac{d}{dz}\,\Phi(z)\right| e^{kw(z)},$$

provided that $x \geq 2,\ k,l \in \mathbb{N}$ and $l \geq 2$. Note that if $x,l \geq 2,\ x \geq (l^d/(l-1))$. Then

$$\left|e^{mw(x)} \int_0^{x+l^d} h_{\alpha,\beta}\,(\psi)\,(y) \int_{|x-y/l|}^{x+y/l} D\,(x,\ y/l,z)\left(\Phi\,(z)-\Phi(x)\right)\right.$$

$$\left.\times \frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma\,(3\alpha+\beta)}\,dz.\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy\right|$$

$$\leq C\, l^{d(10\alpha+6\beta)-1}\,(x+1)^{10\alpha+6\beta}\, e^{mw(x)-kw\,[x-(x/l)-l^{d-1}]},$$

when $x \geq 2, l,\ k \in \mathbb{N}$ *and* $l \geq 2$.

Since w is increasing on $[0,\infty)$ and w verifies the property (i), we have that

$$w\left(x-\frac{x}{l}-l^{d-1}\right) \geq \frac{1}{2}\,w(x)-w(1), \qquad x\geq 2, \qquad l,k \in \mathbb{N} \text{ and } l\geq 2,$$

hence by choosing k large enough, since w satisfies the property (i), we have

$$\left|e^{mw(x)} \int_0^{x+l^d} h_{\alpha,\beta}\,(\psi)\,(y) \int_{|x-y/l|}^{x+y/l} D\,(x,y/l,z)\left(\Phi(z)-\Phi(x)\right)\right.$$

$$\left.\times \frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma\,(3\alpha+\beta)}\,dz\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy\right|$$

$$\leq C\, l^{d(10\alpha+6\beta)-1}, \qquad x\geq 2, \qquad l,k \in \mathbb{N} \text{ and } l\geq 2.$$

Assume now that $0 < d < 1/(10\alpha+6\beta)$. Then we conclude that

$$\sup_{x\geq 2}\left|e^{mw(x)} \int_0^{x+l^d} h_{\alpha,\beta}\,(\psi)\,(y) \int_{|x-y/l|}^{x+y/l} D\,(x,y/l,z)\left(\Phi(z)-\Phi(x)\right)\right.$$

$$\left.\times \frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dz\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\,dy\right| \to 0,\ \text{as } l\to\infty.$$

By proceeding in a similar way we obtain that

$$\underset{0\le x\le 2}{Sup}\left| e^{mw(x)} \int_{0}^{x+l^d} h_{\alpha,\beta}\,(\psi)\,(y) \int_{|x-y/l|}^{x+y/l} D\,(x,y/l,z)\,\left(\Phi(z)-\Phi(x)\right)\right.$$

$$\left. \times \frac{z^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dz\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy \right|$$

$$\le C \underset{z\,\in\,(0,\infty)}{Sup} \left|h_{\alpha,\beta}\,(\psi)\,(z)\right| \frac{1}{l}\,(2+l)^{10\alpha+6\beta} \underset{z\,\in\,(0,\infty)}{Sup}\left|\frac{1}{z}\frac{d}{dz}\,\Phi\,(z)\right| \to 0,\ \ as\ l\to\infty,$$

provided that $0 < d < 1\left(2^{\alpha-\beta}+4\right)$.

Thus, we deduce that

$$B_{m,n}^{\alpha,\beta}\,(\phi_l-\phi)\ \to 0, \qquad as\ l\to\infty.$$

By taking into account Proposition 2.3, the proof is now complete.

Remark 2: According to [2, Corollary 2.8], the Property (ii) (for w) is essential to establish the non-triviality of the space $B_{\alpha,\beta}\,(w)$. Indeed the function $\phi(x) = e^{-x^2/2}, x \in [0,\infty)$ is in $\mathcal{H}_{\alpha,\beta}\,(w)$. (see [8, (10)]) provided that $w(x) \le C\,x^l$, when x is large for some $l < 2$.

Next we establish a result concerning approximated identity in $\mathcal{H}_{\alpha,\beta}\,(w)$ involving Hankel type convolution. This property whose proof will be omitted can be proved following a procedure similar to the one employed to prove [3, Proposition 3.5] and [17].

Proposition 2.7: Assume that $\psi \in B_{\alpha,\beta}\,(w)$ and that $\int_0^\infty \psi\,(x)\,x^{4\alpha}\,dx = 2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)$. Then for every $\phi \in \mathcal{H}_{\alpha,\beta}\,(w),\ \phi\,\#\,\psi_m \to \phi,\ as\ m\to\infty,\ in\ \mathcal{H}_{\alpha,\beta}\,(w)$ where, for each $m \in \mathbb{N},\ \psi_m\,(x) = m^{6\alpha+2\beta}\,\psi\,(mx),\ x \in (0,\infty)$.

Hankel type transformation and Hankel type convolution on the space $\mathcal{H}_{\alpha,\beta}\,(w)'$

In this section we study the Hankel type transformation and the Hankel type convolution on $\mathcal{H}_{\alpha,\beta}\,(w)'$, the dual space of $\mathcal{H}_{\alpha,\beta}\,(w)$. Our results can be seen as an extension of the ones presented in [12].

Suppose that f is a measurable function on $(0,\infty)$ such that, for some $k \in \mathbb{N}$,

$$\int_0^\infty e^{-kw(x)}\,|f(x)|\,x^{4\alpha}\,dx\ <\ \infty,$$

then f defines an element $T_f \in \mathcal{H}_{\alpha,\beta}(w)'$ by

$$\langle T_f, \phi\rangle = \int_0^\infty f(x)\ \phi(x)\ \frac{x^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\, dx, \qquad \phi \in \mathcal{H}_{\alpha,\beta}(w).$$

Indeed, for every $\phi \in \mathcal{H}_{\alpha,\beta}(w)$, it has

$$|\langle T_f, \phi\rangle| \leq C \int_0^\infty e^{-kw(x)}|f(x)|x^{4\alpha}\, dx\ u_{k,0}(\phi).$$

In particular the space $\mathcal{H}_{\alpha,\beta}(w)$ can be identified with a subspace of $\mathcal{H}_{\alpha,\beta}(w)'$.

On the other hand, if $\phi \in \mathcal{H}_{\alpha,\beta}(w)$ then $\phi \in \varepsilon_{\alpha,\beta}(w)$, the space of pointwise multipliers of $B_{\alpha,\beta}(w)$. Indeed, let $\phi \in \mathcal{H}_{\alpha,\beta}(w)$. Assume that $\psi \in B^a_{\alpha,\beta}(w)$ with $a > 0$. Then $\phi(x)\,\psi(x) = 0,\ x \geq a$. Moreover for every $n \in \mathbb{N}$,

$$\delta_n^{\alpha,\beta}(\phi\psi) = \int_0^\infty e^{nw(x)}\left|h_{\alpha,\beta}(\phi\psi)(x)\right| x^{4\alpha}\, dx \leq C\ \delta_n^{\alpha,\beta}(\psi)\, v_{l,0}^{\alpha,\beta}(\phi),$$

where $l \in \mathbb{N}$ is chosen large enough and it is not depending on ϕ. Note that we also have proved that $\mathcal{H}_{\alpha,\beta}(w)$ is continuously contained in $\varepsilon_{\alpha,\beta}(w)$. Hence, the dual space of $\varepsilon_{\alpha,\beta}(w)'$ of $\varepsilon_{\alpha,\beta}(w) \subset \mathcal{H}_{\alpha,\beta}(w)'$.

We define the Hankel type transformation on $\mathcal{H}_{\alpha,\beta}(w)'$ by transposition. That is, if $T \in \mathcal{H}_{\alpha,\beta}(w)'$, the Hankel type transform $h'_{\alpha,\beta}\,T$ of T is the element of $\mathcal{H}_{\alpha,\beta}(w)'$ given through

$$\langle h'_{\alpha,\beta}\,T, \phi\rangle = \langle T, h_{\alpha,\beta}\,\phi\rangle, \qquad \phi \in \mathcal{H}_{\alpha,\beta}(w).$$

The generalized Hankel type transformation $h'_{\alpha,\beta}$ can be seen as an extension of the Hankel type transformation $h_{\alpha,\beta}$. Let $\psi \in \mathcal{H}_{\alpha,\beta}(w)$. Since $h_{\alpha,\beta}(w) \in \mathcal{H}_{\alpha,\beta}(w)$, $h_{\alpha,\beta}(\psi)$ defines an element $T_{h_{\alpha,\beta}(\phi)}$ of $\mathcal{H}_{\alpha,\beta}(w)'$ by

$$\langle T\, h_{\alpha,\beta\,(\psi),}\, \phi\rangle = \int_0^\infty h_{\alpha,\beta}(\psi)(x)\,\phi(x)\,\frac{x^{4\alpha}}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\, dx,\ \ \phi \in \mathcal{H}_{\alpha,\beta}(w).$$

Moreover, Parseval equality for Hankel type transformation leads to

$$\langle T_{h_{\alpha,\beta}}(\psi), \phi\rangle = \int_0^\infty \psi(x)\, h_{\alpha,\beta}(\phi)(x)\,\frac{x^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\, dx,$$

$$= \langle T_\psi,\, h_{\alpha,\beta}(\phi)\rangle, \qquad \phi \in \mathcal{H}_{\alpha,\beta}(w).$$

Thus, we have shown that $T_{h_{\alpha,\beta}(\Psi)} = h'_{\alpha,\beta}\left(T_{\psi}\right)$.

Now, we determine the Hankel type transform of the distributions in $\varepsilon_{\alpha,\beta}(w)'$.

Proposition 3.1: If $T \in \varepsilon_{\alpha,\beta}(w)'$, the Hankel type transform $h'_{\alpha,\beta}\,T$ coincides with the functional defined by the function

$$F(x) = 2^{\alpha-\beta}\Gamma(3\alpha+\beta)\,\langle T(y),(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)\rangle, \qquad x \in (0,\infty).$$

Then $h'_{\alpha,\beta}\,T$ is a continuous function on $[0,\infty)$ and there exist $C > 0$ and $r \in \mathbb{N}$ for which

$$\left|h'_{\alpha,\beta}\,(T)\,(x)\right| \le C\,e^{rw(x)}, \qquad x \in (0,\infty).$$

Proof : Let $T = \varepsilon_{\alpha,\beta}(w)'$. We have to see that

$$\left(h'_{\alpha,\beta}\,(T),\phi\right) = \langle T, h_{\alpha,\beta}\,(\phi)\rangle = \int_0^\infty \langle T(y),(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)\rangle\,\phi\,(x)\,x^{4\alpha}\,dx,$$

for every $\phi \in \mathcal{H}_{\alpha,\beta}\,(w)$. (3.1)

In [2, Proposition 3.4] we proved that, for every $x \in (0,\infty)$, the function f_x defined by $f_x\,(y) = (xy)^{-(x-y)}\,J_{\alpha-\beta}\,(xy)$, $y \in (0,\infty)$ is in $\varepsilon_{\alpha,\beta}(w)$.

Hence, we can define the function

$$F\,(x) = \langle T(y),(xy)^{-(\alpha-\beta)}\,J_{\alpha-\beta}\,(xy)\rangle, \qquad x \in [0,\infty).$$

Thus F is continuous function on $[0,\infty)$. Indeed, let $x_0 \in [0,\infty)$. To See that F is continuous at $x_{0,}$ it is sufficient to show that, for every $n \in \mathbb{N}$ and $\phi \in B_{\alpha,\beta}\,(w)$,

$$\delta_n^{\alpha,\beta}\left(\phi(y)(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy) - (x_0y)^{-(\alpha-\beta)}J_{\alpha-\beta}(x_0y)\right) \to 0, as\ x \to x_0.$$

Assume that $n \in \mathbb{N}$ and $\phi \in B_{\alpha,\beta}\,(w)$. By virtue of [3, (3.4)], it follows for every $x,\ z \in [0,\infty)$,

$$h_{\alpha,\beta}\left(\phi(y)\,(xy)^{-\alpha+\beta}\,J_{\alpha-\beta}(xy) - (x_0y)^{-(\alpha-\beta)}J_{\alpha-\beta}(x_0y)\right)\ (z)$$

$$= \frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\left(\tau_x\left(h_{\alpha,\beta}\phi\right)(z) - \tau_{x_0}\left(h_{\alpha,\beta}\phi\right)(z)\right).$$

According to Proposition 2.4 (ii) and Proposition 2.6, the mapping G defined by

$$G\,(x) = \tau_x\left(h_{\alpha,\beta}\,\phi\right), \qquad x \in [0,\infty),$$

is continuous from $[0,\infty)$ into $\mathcal{H}_{\alpha,\beta}\,(w)$. Moreover, since w satisfies the property (iii), there exists $l \in \mathbb{N}$ such that

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$$\delta_n^{\alpha,\beta}\left(\left(\phi(x)^{-(\alpha-\beta)}J_{\alpha-\beta}(x)-(x_0)^{-(\alpha-\beta)}J_{\alpha-\beta}(x_0)\right)\right)$$

$$=\frac{1}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\int_0^\infty e^{n\,w(z)}\left|\tau_x\left(h_{\alpha,\beta}\,\phi\right)(z)-\tau_{x_0}\left(h_{\alpha,\beta}\,\phi\right)(z)\right|z^{4\alpha}\,dz$$

$$\leq C\,u_{n+l,0}\left(\tau_x\left(h_{\alpha,\beta}\,\phi\right)-\tau_{x_0}\left(h_{\alpha,\beta}\,\phi\right)\right),\qquad x\in[0,\infty).$$

Hence,

$$\delta_n^{\alpha,\beta}\left(\phi(y)\left((xy)^{-\alpha+\beta}J_{\alpha-\beta}(xy)-(x_0y)^{-\alpha+\beta}J_{\alpha-\beta}(x_0y)\right)\right)\to 0,\qquad as\ x\to x_0.$$

Moreover, since $T\in\xi_{\alpha,\beta}(w)'$, there exists $C>0,\ r\in\mathbb{N}$ and $\phi_{1,\ldots\ldots,}\ \phi_r\in B_{\alpha,\beta}\,(w)$,

$$|\langle T,\Phi\rangle|\leq C\max_{j=1,\ldots.,r}\delta_r^{\alpha,\beta}\left(\phi_j\,\Phi\right),\qquad \Phi\in\ \varepsilon_{\alpha,\beta}(w).$$

In particular, since w has the property (iii) for every $x\in(0,\infty)$,

$$\left|\langle T(y),(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}(xy)\rangle\right|\leq C\max_{j=1,2,\ldots,r}\int_0^\infty e^{rw(x)}\left|\tau_x\left(h_{\alpha,\beta}\,\phi_j\right)(y)\right|y^{4\alpha}\,dy$$

$$\leq C\max_{j=1,\ldots,r}u_{r+l,0}\left(\tau_x\left(h_{\alpha,\beta}\,\phi_j\right)\right),$$

for some $l\in\mathbb{N}$. Then by (2.9), it follows that

$$\left|\langle T(y),(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}\,(xy)\rangle\right|\leq C\,e^{(r+l)w(x)}\max_{j=1,\ldots,r}v_{r+l,0}^{\alpha,\beta}\left(\phi_j\right),\ x\in[0,\infty).\qquad(3.2)$$

From (3.2), we infer that the integral in (3.1) is absolutely convergent for every $\phi\in\mathcal{H}_{\alpha,\beta}\,(w)$.

Assume that $\phi\in\mathcal{H}_{\alpha,\beta}\,(w)$, It is clear that

$$\lim_{b\to\infty}\int_b^\infty\langle T(y),\qquad(xy)^{-\alpha+\beta}J_{\alpha-\beta}\,(xy)\rangle\,\phi(x)\,x^{4\alpha}\,dx=0.$$

Let $b>0$. we can write

$$\int_0^\infty\langle T(y),\qquad(xy)^{-(\alpha-\beta)}J_{\alpha-\beta}\,(xy)\rangle\,\phi(x)\,x^{4\alpha}\,dx$$

$$=\lim_{n\to\infty}\langle T(y),\ \frac{b}{n}\sum_{j=1}^n\left(\frac{jb}{n}\,y\right)^{-\alpha+\beta}J_{\alpha-\beta}\left(\frac{jb}{n}\,y\right)\,\phi\,\left(\frac{jb}{n}\right)\left(\frac{jb}{n}\right)^{4\alpha}\rangle\qquad(3.3)$$

We are going to see that

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$$\int_0^b \langle (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, \phi\,(x)\, x^{4\alpha}\, dx$$

$$= \lim_{n\to\infty} \frac{b}{n} \sum_{j=1}^{n} \left(\frac{jb}{n}\, y\right)^{-\alpha+\beta} J_{\alpha-\beta}\left(\frac{jb}{n}\, y\right) \phi\left(\frac{jb}{n}\right) \left(\frac{jb}{n}\right)^{4\alpha} \rangle.$$

We are going to see that

$$\int_0^b (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}(xy)\, \phi(x)\, x^{4\alpha}\, dx$$

$$= \lim_{n\to\infty} \frac{b}{n} \sum_{j=1}^{n} \left(\frac{jb}{n} y\right)^{-(\alpha-\beta)} J_{\alpha-\beta}\left(\frac{jb}{n} y\right) \phi\left(\frac{jb}{n}\right) \left(\frac{jb}{n}\right)^{4\alpha},$$

in the sense of convergence of $\varepsilon_{\alpha,\beta}\,(w)$.

Indeed, let $\psi \in B_{\alpha,\beta}\,(w)$ and $m \in \mathbb{N}$. It has, for some $l \in \mathbb{N}$,

$$\delta_m^{\alpha,\beta}\left(\psi(y)\left(\int_0^b (xy)^{-\alpha+\beta}\, J_{\alpha-\beta}\,(xy)\, \phi(x)\, x^{4\alpha}\, dx\right.\right.$$

$$\left.\left. - \frac{b}{n} \sum_{j=1}^{n} \left(\frac{jb}{n}\, y\right)^{-\alpha+\beta} J_{\alpha-\beta}\left(\frac{jb}{n}\, y\right) \phi\left(\frac{jb}{n}\right) \left(\frac{jb}{n}\right)^{4\alpha}\right)\right)$$

$$\leq C\, u_{l,0}\left(h_{\alpha,\beta}\left(\psi(y)\left(\int_0^b (xy)^{-(\alpha-\beta)} J_{\alpha-\beta}\,(xy)\, \phi(x)\, x^{4\alpha}\, dx\right.\right.\right.$$

$$\left.\left.\left. - \frac{b}{n} \sum_{j=1}^{n} \left(\frac{jb}{n} y\right)^{-\alpha+\beta} J_{\alpha-\beta}\left(\frac{jb}{n}\, y\right) \phi\left(\frac{jb}{n}\right) \left(\frac{jb}{n}\right)^{4\alpha}\right)\right)\right)$$

$$\leq C\, u_{l,0} \begin{pmatrix} \int_0^b \phi(x)\, x^{4\alpha}\, \tau_x\big(h_{\alpha,\beta}\, \psi\big)\,(z)\, dx \\ - \frac{b}{n} \sum_{j=1}^{n} \phi\left(\frac{jb}{n}\right) \left(\frac{jb}{n}\right)^{4\alpha} \tau_{jb/n}\left(h_{\alpha,\beta}\, \psi\right)(z) \end{pmatrix}.$$

Note that from (2.9), it follows that

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$$e^{lw(z)}\left|\int_0^b \phi(x)\, x^{4\alpha}\, \tau_x\left(h_{\alpha,\beta}\,\psi\right)(z)\, dx - \frac{b}{n}\sum_{j=1}^{n}\phi\left(\frac{jb}{n}\right)\left(\frac{jb}{n}\right)^{4\alpha}\ \tau_{jb/n}\left(h_{\alpha,\beta}\,\psi\right)(z)\right|$$

$$\leq C\, e^{-w(z)}\left(\begin{array}{c}\displaystyle\int_0^b |\phi(x)|\, x^{4\alpha}\, e^{(l+1)w(x)}\, dx \\ \displaystyle+\frac{b}{n}\sum_{j=1}^{n}\left|\phi\left(\frac{jb}{n}\right)\right|\left(\frac{jb}{n}\right)^{4\alpha} e^{(l+1)w(jb/n)}\end{array}\right)$$

$$\leq C\, e^{-w(z)}, \qquad z\ \in\ (0,\infty).$$

Hence, if $\epsilon > 0$, there exists $z_0\ \in\ (0,\infty)$ such that

$$\operatorname*{Sup}_{z\geq z_0} e^{lw(z)}\left|\int_0^b \phi(x)\, x^{4\alpha}\, \tau_x\left(h_{\alpha,\beta}\,\psi\right)(z)\, dx - \frac{b}{n}\ \sum_{j=1}^{n}\phi\ \left(\frac{jb}{n}\right)\left(\frac{jb}{n}\right)^{4\alpha}\ \tau_{jb/n}\left(h_{\alpha,\beta}\,\psi\right)(z)\right| < \epsilon.$$

On the other hand, since the function H defined by

$$H\,(x,z) =\ \phi\,(x)\, x^{4\alpha}\, \tau_x\left(h_{\alpha,\beta}\,\psi\right)(z), \qquad x, \qquad z\ \in [0,\infty),$$

is uniformly continuous in $(x,z)\ \in [0,b]\times[0,z_0]$, it has

$$\lim_{n\to\infty}\frac{b}{n}\sum_{j=1}^{n}\phi\ \left(\frac{jb}{n}\right)\left(\frac{jb}{n}\right)^{4\alpha}\ \tau_x\left(h_{\alpha,\beta}\,\psi\right)\left(\frac{jb}{n}\right)$$

$$= \int_0^b \phi\,(x)\, x^{4\alpha}\, \tau_x\left(h_{\alpha,\beta}\,\psi\right)(x)\, dx,$$

uniformly in $[0,x_0]$.

From the above arguments we conclude (3.4) in the sense of convergence in $\varepsilon_{\alpha,\beta}\,(w)$. Hence it has that

$$\int_0^b \langle T(y),(xy)^{-\alpha+\beta}J_{\alpha-\beta}(xy)\rangle\, \phi(x)x^{4\alpha}\, dx = \langle T(y),\int_0^b (xy)^{-\alpha+\beta}J_{\alpha-\beta}(xy)\,\phi(x)\, x^{4\alpha}dx\rangle.$$

Also,

$$\lim_{b\to\infty}\int_b^\infty (xy)^{-\alpha+\beta}\, J_{\alpha-\beta}\,(xy)\ \phi(x)\, x^{4\alpha}\, dx = 0$$

in the sense of convergence in $\varepsilon_{\alpha,\beta}\,(w)$.

Indeed, assume that $b > 0$, $\psi \in B_{\alpha,\beta}(w)$ and $m \in \mathbb{N}$. For a certain $l \in \mathbb{N}$ we have that

$$\delta_m^{\alpha,\beta}\left(\left(\psi(y)\int_b^\infty (xy)^{-\alpha+\beta} J_{\alpha-\beta}(xy)\,\phi(x)\,x^{4\alpha}\,dx\right)\right)$$

$$\le C\,u_{l,0}\left(h_{\alpha,\beta}\left(\psi(y)\int_b^\infty (xy)^{-\alpha+\beta} J_{\alpha-\beta}(xy)\,\phi(x)\,x^{4\alpha}\,dx\right)\right)$$

$$\le C \underset{z\,\in\,(0,\infty)}{Sup}\, e^{lw(z)}\left|\int_b^\infty \phi(x)\,\tau_x\left(h_{\alpha,\beta}\,\psi\right)(x)\,x^{4\alpha}\,dx\right|$$

$$\le C\int_b^\infty \left(\phi(x)\left|e^{lw(x)}\,x^{4\alpha}\,dx\,v_{l,0}^{\alpha,\beta}(\psi)\right.\right).$$

Hence,

$$\lim_{b\to\infty}\delta_m^{\alpha,\beta}\left(\psi(y)\int_b^\infty (xy)^{-\alpha+\beta} J_{\alpha-\beta}(xy)\,\phi(x)\,x^{4\alpha}\,dx\right) = 0.$$

Now, standard arguments allow us to show that (3.1) holds.

Thus proof is completed.

Proposition 2.4 (i) allows us to define the Hankel type convolution $T\,\#\,\phi$ of $T \in \mathcal{H}_{\alpha,\beta}(w)'$ and $\phi \in \mathcal{H}_{\alpha,\beta}(w)$ as follows

$$(T\#\phi)(x) = \langle T, \tau_x\,\phi\rangle, \qquad x \in [0,\infty).$$

Note that the last definition extends the Hankel type convolution from $\mathcal{H}_{\alpha,\beta}(w) \times \mathcal{H}_{\alpha,\beta}(w)$ to $\mathcal{H}_{\alpha,\beta}(w)' \times \mathcal{H}_{\alpha,\beta}(w)$. Indeed, let $\phi,\ \psi \in \mathcal{H}_{\alpha,\beta}(w)$.

We can write

$$\left(T_\phi\,\#\,\psi\right)(x) = \langle T_\phi, \tau_x\,\psi\rangle = \int_0^\infty \phi(y)\,(\tau_x\psi)(y)\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy$$

$$= (\phi\,\#\,\psi)(x), \qquad x \in [0,\infty).$$

We now prove that $T\,\#\,\phi \in \mathcal{H}_{\alpha,\beta}(w)'$ for every $T \in \mathcal{H}_{\alpha,\beta}(w)'$ and $\phi \in \mathcal{H}_{\alpha,\beta}(w)$.

Proposition 3.2: Let $T \in \mathcal{H}_{\alpha,\beta}(w)$ and $\phi \in \mathcal{H}_{\alpha,\beta}(w)$. Then $T\#\,\phi$ is a continuous function on $[0,\infty)$. Moreover, there exist $C > 0$ and $r \in \mathbb{N}$ such that

$$|(T\#\phi)\ (x)| \leq C\ e^{rw(x)}, \qquad x\ \in\ [0,\infty).$$

Hence, $T\#\phi$ defines an element of $\mathcal{H}_{\alpha,\beta}\ (w)'$.

Proof: By Proposition 2.4 (ii), $T\#\phi$ is a continuous function on $[0,\infty)$.

Further, since $T\ \in\ \mathcal{H}_{\alpha,\beta}\ (w)'$, from Proposition 2.3 it implies that there exist $C\ > 0$ and $r\ \in\ \mathbb{N}$ such that

$$|\langle T,\psi\rangle|\ \leq C\ \max_{0\leq n\leq r}\left\{A_{r,n}^{\alpha,\beta}\ (\psi),\ \ v_{r,n}^{\alpha,\beta}(\psi)\right\}, \qquad \psi\ \in\ \mathcal{H}_{\alpha,\beta}\ (w).$$

In particular, we have

$$|(T\#\phi)\ (x)|\ \leq C\ \max_{0\leq n\leq r}\left\{A_{r,n}^{\alpha,\beta}\ (\tau_x\ \phi),\ v_{r,n}^{\alpha,\beta}(\tau_x\phi)\right\},\ \ x\ \in\ [0,\infty).$$

From (2.9), it is deduced that,

$$A_{r,n}^{\alpha,\beta}\ (\tau_x\phi)\ \leq\ e^{rw(x)}\ A_{r,n}^{\alpha,\beta}\ (\phi), \qquad x\ \in\ [0,\infty)\ and\ n\ \in\ \mathbb{N}\,.$$

Also (2.10) implies, since w satisfies the property (c), that

$$v_{r,n}^{\alpha,\beta}\ (\tau_x\phi)\ \leq C\ (1+x^{2n})\sum_{j=0}^{n} v_{r,n}^{\alpha,\beta}(\phi)$$

$$\leq C\ e^{lw(x)}\ \sum_{j=0}^{n} v_{r,j}^{\alpha,\beta}\ (\phi), \qquad x\ \in\ [0,\infty)\ \ and\ \ r\ \in\ \mathbb{N},$$

for some $l\ \in\ \mathbb{N}$.

Hence, for a certain $m\ \in\ \mathbb{N}$,

$$|(T\#\phi)\ (x)|\ \leq C\ e^{mw(x)}\,, \qquad x\ \in\ [0,\infty).$$

Thus proof is completed.

Now, we introduce, for every $m\ \in\ \mathbb{N}$, the space $\mathcal{A}_m\ (w)$ constituted by all those functions f defined on $(0,\infty)$ such that

$$\sup_{x\,\in(0,\infty)}\ e^{-mw(x)}\ |f(x)|\ <\ \infty.$$

A careful reading of the proof of Proposition 3.2 allows us to deduce that if $\tau\ \in\ \mathcal{H}_{\alpha,\beta}\ (w)'$, there exists $r\ \in\ \mathbb{N}$ such that $T\ \#\ \phi\ \ \in\ \mathcal{A}_r\ (w)$ for every $\phi\ \in\ \mathcal{H}_{\alpha,\beta}\ (w)$.

Now we establish an associative property for the distributional convolution.

Proposition 3.3: Let $\tau\ \in\ \mathcal{H}_{\alpha,\beta}\ (w)'$, and $\phi,\ \psi\ \in\ \mathcal{H}_{\alpha,\beta}\ (w)$. Then

$$(T\#\phi)\ \#\ \psi = T\#(\phi\#\psi). \qquad (3.5)$$

Proof: Following Proposition 3.2, $T\#\phi$ defines an element of $\mathcal{H}_{\alpha,\beta}\ (w)'$ and we have

$$\left((T\#\phi)\#\,\psi\right)(x) = \int_0^\infty (T\#\phi)\ (y)\,(\tau_x w)\,(y)\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy$$

$$= \int_0^\infty \langle T,\tau_y\phi\rangle\,(\tau_x\psi)\,(y)\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy\,,\ \ x\ \in\ (0,\infty)\,.$$

Equality (3.5) will be proved when we see that, for every $x\ \in\ (0,\infty)$,

$$\int_0^\infty \langle T,\tau_y\,\phi\rangle\ (\tau_x\psi)\,(y)\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy$$

$$= \langle T(z),\int_0^\infty (\tau_x\,\phi)\,(z)\,(\tau_x\psi)\,(y)\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\ \Gamma\,(3\alpha+\beta)}\ dy\rangle\,.$$

We have (3.6)

$$\int_0^\infty \left(\tau_y\,\phi\right)(z)\left(\tau_y\,\psi\right)(y)\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy$$

$$= \int_0^\infty (\tau_z\,\phi)\ (y)\,(\tau_x\,\psi)\,(y)\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy$$

$$= (\tau_x\,\phi\,\#\,\psi)\,(x) = \tau_x(\phi\#\psi)(z), \qquad x,z,\in[0,\infty).$$

Our objective is to prove (3.6). We will use a procedure similar to the one employed in the proof of Proposition 3.1.

Let $x\ \in\ [0,\infty)$. By virtue of Proposition 3.2, it follows that

$$\lim_{b\to\infty}\int_b^\infty \langle T,\tau_y\phi\rangle\ (\tau_x\,\psi)\,(y)\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy = 0\,. \qquad (3.7)$$

Assume that $m,n\ \in\ \mathbb{N}$. According to (2.9), we can write

$$A^{\alpha,\beta}_{m,n}\left(\int_0^\infty (\tau_x\phi)\,(y)(\tau_x\psi)\,(y)\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy\right)$$

$$\le \int_0^\infty e^{mw(y)}\ |(\tau_x\psi)\,(y)|\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy\ A^{\alpha,\beta}_{a_{n,n}}\,(\phi),\quad b>0.$$

Thus from Proposition 2.4 (i), it is inferred that

$$\lim_{b\to\infty} A^{\alpha,\beta}_{m,n}\left(\int_b^\infty (\tau_x\phi)\,(y)\,(\tau_x\psi)(y)\ \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\ dy\right) = 0.$$

On the other hand, for every $b > 0$,

$$\left(\frac{1}{t} D\right)^{n} h_{\alpha,\beta} \left(\int_{b}^{\infty} (\tau_z \phi)(y)(\tau_x \psi)(y) \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} dy \right)(t)$$

$$= \sum_{j=0}^{n} (-1)^j \binom{n}{j} \int_{b}^{\infty} (\tau_x \phi)(y)\, y^{2j} (yt)^{-\alpha+\beta-j} J_{\alpha-\beta+j}(yt)\, y^{4\alpha}\, dy$$

$$\times \left(\frac{1}{t} D\right)^{n-j} h_{\alpha,\beta}(\phi)(t), \qquad t \in (0,\infty).$$

Thus, by Proposition 2.4(i) and taking into account the boundedness of the function $z^{-\alpha+\beta} J_{\alpha-\beta}(z)$ on $(0,\infty)$, we have

$$v_{m,n}^{\alpha,\beta} \left(\int_{b}^{\infty} (\tau_z \phi)(y)\, (\tau_x \psi)(y) \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} dy \right)$$

$$\leq C \sum_{j=0}^{n} v_{m,n-j}^{\alpha,\beta}(\phi) \int_{b}^{\infty} |(\tau_x \psi)(y)|\, y^{2j+4\alpha}\, dy \to 0, \qquad as\ b \to \infty.$$

Therefore, we see that

$$\int_b^{\infty} (\tau_z \phi)(y)\,(\tau_x \psi)(y) \frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)} dy \to 0, \ \ as\ b \to \infty, \tag{3.8}$$

in the sense of convergence in $\mathcal{H}_{\alpha,\beta}(w)$.

Let $b > 0$. By using, as in the proof of proposition 3.1, Riemann sums, we can prove that

$$\int_0^b \langle T, \tau_y \phi \rangle (\tau_x \psi)(y)\, y^{4\alpha}\, dy = \langle T(z), \int_0^b (\tau_y \phi)(z)\, (\tau_x \psi)(y)\, y^{4\alpha}\, dy \rangle. \tag{3.9}$$

Thus by combining (3.7), (3.8) and (3.9), we deduce (3.6) and therefore proof of (3.5) is completed.

As a special case, we have following corollary.

Corollary 3.4: Let $T \in \mathcal{H}_{\alpha,\beta}(w)'$ and $\phi,\ \psi \in h_{\alpha,\beta}(w)$. Then

$$\langle T \# \phi, \psi \rangle = \langle T, \phi \# \psi \rangle. \tag{3.10}$$

Proof: To see (3.10), it is sufficient to take $x = 0$ in (3.5).

Remark 3: Note that the property in Corollary 3.4 is equivalent to the one in Proposition 3.3. Indeed, let $T \in \mathcal{H}_{\alpha,\beta}(w)'$ and $\phi, \psi \in \mathcal{H}_{\alpha,\beta}(w)$.

If $x \in [0,\infty)$, $\tau_x \psi \in \mathcal{H}_{\alpha,\beta}(w)$ (Proposition 2.4 (i)). Then from Corollary 3.4 we deduce

$$
\begin{aligned}
(T\#\phi)\#\,\psi)\,(x) &= \langle T, \phi\,\#\,(\tau_x\psi)\rangle \\
&= \langle T, \tau_x\,(\phi\,\#\,\psi)\rangle \\
&= \big(T\,\#\,(\phi\,\#\,\psi)\big)\,(x), \qquad x\ \in [0,\infty).
\end{aligned}
$$

Thus Proposition 3.3 is established.

Now we obtain a distributional version of the interchange formula.

Proposition 3.5: Let $T\ \in \mathcal{H}_{\alpha,\beta}(w)'$ and $\phi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$. Then

$$h'_{\alpha,\beta}\,(T\#\phi) = h'_{\alpha,\beta}\,(T)\,h_{\alpha,\beta}\,(\phi).$$

Proof : Assume that $\psi\ \in\ \mathcal{H}_{\alpha,\beta}\,(w)$. According to Corollary 3.4, we can write

$$
\begin{aligned}
\langle h'_{\alpha,\beta}\,(T\#\phi),\quad \psi\rangle &= \langle T\#\phi,\, h_{\alpha,\beta}\,(\psi)\rangle = \langle T,\ \ \phi\#\, h_{\alpha,\beta}\,(\psi)\rangle \\
&= \langle T,\quad h_{\alpha,\beta}\,\big(h_{\alpha,\beta}\,(\phi)\,\psi\big)\rangle = \langle h'_{\alpha,\beta}\,(T)\,h_{\alpha,\beta}\,(\phi),\psi\rangle.
\end{aligned}
$$

Thus proof is completed.

Another consequence of Corollary 3.4 is the following.

Proposition 3.6: The space

$$\mathcal{A}\,(w) = \bigcup_{m\,\in\,\mathbb{N}} \mathcal{A}_m\,(w)$$

is a weak * dense subspace of $\mathcal{H}_{\alpha,\beta}(w)'$.

Proof: It is sufficient to take into account the remark after Proposition 3.2 and to use Proposition 2.7 and Corollary 3.4.

We now introduce the space $\mathcal{F}_{\alpha,\beta}\,(w)$ that consists of all those $T\ \in\ B_{\alpha,\beta}\,(w)'$ for which there exists a function G_T belonging to $\mathcal{A}_m\,(w)$ for some $m\ \in\ \mathbb{N}$ such that

$$\langle T,\phi\rangle = \int_0^\infty G_T\,(y)\,h_{\alpha,\beta}\,(\phi)\,(y)\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy,\ \ \phi\ \in\ B_{\alpha,\beta}\,(w). \tag{3.11}$$

Note that the right hand side of (3.11) defines a continuous functional on $\mathcal{H}_{\alpha,\beta}\,(w)$. Hence T can be extended to $\mathcal{H}_{\alpha,\beta}\,(w)$ as an element of $\mathcal{H}_{\alpha,\beta}\,(w)'$. We denote by T that extension to $\mathcal{H}_{\alpha,\beta}\,(w)$. Moreover, for every $\phi\ \in\ \mathcal{H}_{\alpha\beta}\,(w)$, it has

$$
\begin{aligned}
\langle h'_{\alpha,\beta}\,T,\phi\rangle\ &= \langle T, h_{\alpha,\beta}\,(\phi)\rangle \\
&= \int_0^\infty G_T\,(y)\ h_{\alpha,\beta}\,\big(h_{\alpha,\beta}(\phi)\big)\,(y)\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\Gamma(3\alpha+\beta)}\,dy
\end{aligned}
$$

$$= \int_0^\infty G_T(y)\,\phi(y)\,\frac{y^{4\alpha}}{2^{\alpha-\beta}\,\Gamma(3\alpha+\beta)}\,dy.$$

Hence $h'_{\alpha,\beta}\,T$ coincides with the functional generated by G_T on $\mathcal{H}_{\alpha,\beta}\,(w)'$.

We can also prove that if $T\in\mathcal{F}_{\alpha,\beta}\,(w)$ and $\phi\in\mathcal{H}_{\alpha,\beta}\,(w)$, then $T\#\phi$ and $T.\phi$ are in $\mathcal{F}_{\alpha,\beta}\,(w)$.

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